SECTION 1 (Maximum Marks: 18)

- This section contains **SIX (06)** questions.
- Each question has **FOUR** options. **ONLY ONE** of these four options is the correct answer.
- For each question, choose the option corresponding to the correct answer.
- Answer to each question will be evaluated according to the following marking scheme:

Full Marks : +3 If ONLY the correct option is chosen;
Zero Marks : 0 If none of the options is chosen (i.e. the question is unanswered);
Negative Marks : −1 In all other cases.

Q.1 Suppose a, b denote the distinct real roots of the quadratic polynomial $x^2 + 20x - 2020$ and suppose c, d denote the distinct complex roots of the quadratic polynomial $x^2 - 20x + 2020$. Then the value of

$$ac(a-c) + ad(a-d) + bc(b-c) + bd(b-d)$$

is

(A) 0 (B) 8000 (C) 8080 (D) 16000

Q.2 If the function $f: \mathbb{R} \longrightarrow \mathbb{R}$ is defined by $f(x) = |x|(x - \sin x)$, then which of the following statements is **TRUE**?

(A) f is one-one, but **NOT** onto

(B) f is onto, but **NOT** one-one

(C) f is **BOTH** one-one and onto

(D) f is **NEITHER** one-one **NOR** onto

Q.3 Let the functions $f: \mathbb{R} \longrightarrow \mathbb{R}$ and $g: \mathbb{R} \longrightarrow \mathbb{R}$ be defined by

$$f(x) = e^{x-1} - e^{-|x-1|} \quad \text{and} \quad g(x) = \frac{1}{2}\left(e^{x-1} + e^{1-x}\right).$$

Then the area of the region in the first quadrant bounded by the curves $y = f(x)$, $y = g(x)$ and $x = 0$ is

(A) $\left(2 - \sqrt{3}\right) + \frac{1}{2}\left(e - e^{-1}\right)$ (B) $\left(2 + \sqrt{3}\right) + \frac{1}{2}\left(e - e^{-1}\right)$

(C) $\left(2 - \sqrt{3}\right) + \frac{1}{2}\left(e + e^{-1}\right)$ (D) $\left(2 + \sqrt{3}\right) + \frac{1}{2}\left(e + e^{-1}\right)$

Q.4 Let a, b and λ be positive real numbers. Suppose P is an end point of the latus rectum of the parabola $y^2 = 4\lambda x$, and suppose the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ passes through the point P. If the tangents to the parabola and the ellipse at the point P are perpendicular to each other, then the eccentricity of the ellipse is

(A) $\frac{1}{\sqrt{2}}$ (B) $\frac{1}{2}$ (C) $\frac{1}{3}$ (D) $\frac{2}{5}$

Q.5 Let C_1 and C_2 be two biased coins such that the probabilities of getting head in a single toss are $\frac{2}{3}$ and $\frac{1}{3}$, respectively. Suppose α is the number of heads that appear when C_1 is tossed twice, independently, and suppose β is the number of heads that appear when C_2 is tossed twice, independently. Then the probability that the roots of the quadratic polynomial $x^2 - \alpha x + \beta$ are real and equal, is

(A) $\frac{40}{81}$ (B) $\frac{20}{81}$ (C) $\frac{1}{2}$ (D) $\frac{1}{4}$

Q.6 Consider all rectangles lying in the region

$$\left\{(x, y) \in \mathbb{R} \times \mathbb{R} : 0 \leq x \leq \frac{\pi}{2} \text{ and } 0 \leq y \leq 2\sin(2x)\right\}$$

and having one side on the x-axis. The area of the rectangle which has the maximum perimeter among all such rectangles, is

(A) $\frac{3\pi}{2}$ (B) π (C) $\frac{\pi}{2\sqrt{3}}$ (D) $\frac{\pi\sqrt{3}}{2}$

SECTION 2 (Maximum Marks: 24)

- This section contains **SIX (06)** questions.
- Each question has **FOUR** options. **ONE OR MORE THAN ONE** of these four option(s) is(are) correct answer(s).
- For each question, choose the option(s) corresponding to (all) the correct answer(s).
- Answer to each question will be evaluated according to the following marking scheme:

Full Marks : +4 If only (all) the correct option(s) is(are) chosen;
Partial Marks : +3 If all the four options are correct but ONLY three options are chosen;
Partial Marks : +2 If three or more options are correct but ONLY two options are chosen, both of which are correct;
Partial Marks : +1 If two or more options are correct but ONLY one option is chosen and it is a correct option;
Zero Marks : 0 If none of the options is chosen (i.e. the question is unanswered);
Negative Marks : −2 In all other cases.

Q.7 Let the function $f: \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^3 - x^2 + (x-1)\sin x$ and let $g: \mathbb{R} \to \mathbb{R}$ be an arbitrary function. Let $fg: \mathbb{R} \to \mathbb{R}$ be the product function defined by $(fg)(x) = f(x)g(x)$. Then which of the following statements is/are TRUE?

(A) If g is continuous at $x = 1$, then fg is differentiable at $x = 1$

(B) If fg is differentiable at $x = 1$, then g is continuous at $x = 1$

(C) If g is differentiable at $x = 1$, then fg is differentiable at $x = 1$

(D) If fg is differentiable at $x = 1$, then g is differentiable at $x = 1$

Q.8 Let M be a 3×3 invertible matrix with real entries and let I denote the 3×3 identity matrix. If $M^{-1} = \text{adj}\,(\text{adj}\,M)$, then which of the following statements is/are ALWAYS TRUE?

(A) $M = I$ (B) $\det M = 1$ (C) $M^2 = I$ (D) $(\text{adj}\,M)^2 = I$

Q.9 Let S be the set of all complex numbers z satisfying $|z^2 + z + 1| = 1$. Then which of the following statements is/are TRUE?

(A) $\left|z + \frac{1}{2}\right| \le \frac{1}{2}$ for all $z \in S$

(B) $|z| \le 2$ for all $z \in S$

(C) $\left|z + \frac{1}{2}\right| \ge \frac{1}{2}$ for all $z \in S$

(D) The set S has exactly four elements

Q.10 Let x, y and z be positive real numbers. Suppose x, y and z are the lengths of the sides of a triangle opposite to its angles X, Y and Z, respectively. If

$$\tan\frac{X}{2} + \tan\frac{Z}{2} = \frac{2y}{x+y+z},$$

then which of the following statements is/are TRUE?

(A) $2Y = X + Z$ (B) $Y = X + Z$

(C) $\tan\frac{X}{2} = \frac{x}{y+z}$ (D) $x^2 + z^2 - y^2 = xz$

Q.11 Let L_1 and L_2 be the following straight lines.

$$L_1: \frac{x-1}{1} = \frac{y}{-1} = \frac{z-1}{3} \quad \text{and} \quad L_2: \frac{x-1}{-3} = \frac{y}{-1} = \frac{z-1}{1}.$$

Suppose the straight line

$$L: \frac{x-\alpha}{l} = \frac{y-1}{m} = \frac{z-\gamma}{-2}$$

lies in the plane containing L_1 and L_2, and passes through the point of intersection of L_1 and L_2. If the line L bisects the acute angle between the lines L_1 and L_2, then which of the following statements is/are TRUE?

(A) $\alpha - \gamma = 3$ (B) $l + m = 2$ (C) $\alpha - \gamma = 1$ (D) $l + m = 0$

Q.12 Which of the following inequalities is/are TRUE?

(A) $\int_0^1 x \cos x \; dx \geq \frac{3}{8}$

(B) $\int_0^1 x \sin x \; dx \geq \frac{3}{10}$

(C) $\int_0^1 x^2 \cos x \, dx \geq \frac{1}{2}$

(D) $\int_0^1 x^2 \sin x \, dx \geq \frac{2}{9}$

SECTION 3 (Maximum Marks: 24)

- This section contains **SIX (06)** questions. The answer to each question is a **NUMERICAL VALUE.**
- For each question, enter the correct numerical value of the answer using the mouse and the on-screen virtual numeric keypad in the place designated to enter the answer. If the numerical value has more than two decimal places, **truncate/round-off** the value to **TWO** decimal places.
- Answer to each question will be evaluated according to the following marking scheme:
  Full Marks : +4 If ONLY the correct numerical value is entered;
  Zero Marks : 0 In all other cases.

Q.13 Let m be the minimum possible value of $\log_3(3^{y_1} + 3^{y_2} + 3^{y_3})$, where y_1, y_2, y_3 are real numbers for which $y_1 + y_2 + y_3 = 9$. Let M be the maximum possible value of $(\log_3 x_1 + \log_3 x_2 + \log_3 x_3)$, where x_1, x_2, x_3 are positive real numbers for which $x_1 + x_2 + x_3 = 9$. Then the value of $\log_2(m^3) + \log_3(M^2)$ is _____

Q.14 Let $a_1, a_2, a_3, \ldots$ be a sequence of positive integers in arithmetic progression with common difference 2. Also, let $b_1, b_2, b_3, \ldots$ be a sequence of positive integers in geometric progression with common ratio 2. If $a_1 = b_1 = c$, then the number of all possible values of c, for which the equality

$$2(a_1 + a_2 + \cdots + a_n) = b_1 + b_2 + \cdots + b_n$$

holds for some positive integer n, is _____

Q.15 Let $f: [0, 2] \longrightarrow \mathbb{R}$ be the function defined by

$$f(x) = (3 - \sin(2\pi x)) \sin\left(\pi x - \frac{\pi}{4}\right) - \sin\left(3\pi x + \frac{\pi}{4}\right).$$

If $\alpha, \beta \in [0, 2]$ are such that $\{x \in [0, 2] : f(x) \geq 0\} = [\alpha, \beta]$, then the value of $\beta - \alpha$ is _____

Q.16 In a triangle PQR, let $\vec{a} = \overrightarrow{QR}$, $\vec{b} = \overrightarrow{RP}$ and $\vec{c} = \overrightarrow{PQ}$. If

$$|\vec{a}| = 3, \qquad |\vec{b}| = 4 \quad \text{and} \quad \frac{\vec{a} \cdot (\vec{c} - \vec{b})}{\vec{c} \cdot (\vec{a} - \vec{b})} = \frac{|\vec{a}|}{|\vec{a}| + |\vec{b}|},$$

then the value of $|\vec{a} \times \vec{b}|^2$ is _____

Q.17 For a polynomial $g(x)$ with real coefficients, let m_g denote the number of distinct real roots of $g(x)$. Suppose S is the set of polynomials with real coefficients defined by

$$S = \{(x^2 - 1)^2(a_0 + a_1 x + a_2 x^2 + a_3 x^3) : a_0, a_1, a_2, a_3 \in \mathbb{R}\}.$$

For a polynomial f, let f' and f'' denote its first and second order derivatives, respectively. Then the minimum possible value of $(m_{f'} + m_{f''})$, where $f \in S$, is _____

Q.18 Let e denote the base of the natural logarithm. The value of the real number a for which the right hand limit

$$\lim_{x \to 0^+} \frac{(1 - x)^{\frac{1}{x}} - e^{-1}}{x^a}$$

is equal to a nonzero real number, is _____

END OF THE QUESTION PAPER